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Zero modes of the $SU(2)_k$ Wess–Zumino–Novikov–Witten model in Euler angles parametrization

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Abstract

We derive the Poisson brackets of the $SU(2)_k$ Wess–Zumino–Novikov–Witten chiral zero modes directly, using Euler angles parametrization.

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1. Introduction

A classical mechanical model of the so-called *zero modes* of the chiral $SU(n)_k$ Wess–Zumino–Novikov–Witten (WZNW) model has been canonically treated in [1]. The symplectic form for the chiral WZNW zero modes appeared first in [2, 3] and later in [4]. These are chiral versions of what has been called the (generalized) ‘top’ in [5, 6].

The zero modes a_C , $C = L, R$, are quantities of zero energy that keep track of the freedom appearing in the standard decomposition of the two-dimensional WZNW field $g(x, t)$ into a product of left- and right-moving chiral parts,

$$g(x, t) = g_L(x^+)g_R^{-1}(x^-) \quad x^\pm := x \pm t \quad g_C(x) = u_C(x)a_C \quad (1.1)$$

so that

$$g(x, 0) = u_L(x)Qu_R^{-1}(x) \quad Q := a_L a_R^{-1}. \quad (1.2)$$

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Classically, $g(x, t) \in G$ where G is a (compact, simple) Lie group, and (1.1) is the general solution of the equations of motion [7]

$$\begin{aligned} \partial_- j_L(x, t) = 0 &\Leftrightarrow \partial_+ j_R(x, t) = 0 & \partial_{\pm} &:= \frac{\partial}{\partial x^{\pm}} \\ j_L = \frac{k}{2\pi i} (\partial_+ g) g^{-1} & & j_R = \frac{k}{2\pi i} g^{-1} \partial_- g & & k \in \mathbb{N}. \end{aligned} \quad (1.3)$$

It is worth mentioning here that $a_C, C = L, R$, can also be interpreted as intertwiners between different initial conditions of the ‘classical KZ equations’ for the chiral fields (see, e.g., [5]):

$$k\partial_x g_L(x) = 2\pi i j_L(x, 0) g_L(x) \quad k\partial_x g_R(x) = -2\pi i j_R(x, 0) g_R(x). \quad (1.4)$$

The WZNW model being (globally) conformal invariant, it is assumed that $g(x + 2\pi, t) = g(x, t)$ (the space is compact, and t is the conformal time). The chiral fields $g_C(x)$ have general group-valued monodromies M_C whereas the monodromies $(M_p)_C$ of $u_C(x)$ are ‘diagonal’, i.e. belong to the maximal torus $T_G \in G$:

$$g_C(x + 2\pi) = g_C(x) M_C \quad u_C(x + 2\pi) = u_C(x) (M_p)_C \quad (1.5)$$

so that

$$M_C = a_C^{-1} (M_p)_C a_C. \quad (1.6)$$

In order for $g(x, t)$ (1.1) to be periodic in x , one should impose the constraint $M_L = M_R$. The relation between $g_C(x)$ and $u_C(x)$ displayed in (1.1) (i.e. between chiral fields with general monodromy and those with diagonal one) is sometimes called ‘vertex-IRF correspondence’. All this is well known; for more details see, e.g., the recent papers [8–10] and references therein.

Both $g_C(x)$ and $u_C(x)$ have quadratic Poisson brackets (PB) involving classical r -matrices. The r -matrix appearing in the PB of $u_C(x)$ [11] (denoted in this paper as $r(p)$) is necessarily *dynamical* [12, 13] whereas, as advocated by Gawędzki in [14], that in the PB of $g_C(x)$ can be chosen to be a *constant* one (see [8, 15] for a general treatment of the problem). The advantage of such a choice is that, after quantization, it provides simple quantum group symmetric exchange relations.

As can be foreseen from the relation between $g_C(x)$ and $u_C(x)$ in (1.1), the exchange relations of the quantized chiral zero modes a_C involve in this case both the (quantum) *dynamical* and *constant* R -matrices [10, 13, 16]. Taking the quasiclassical limit, one should be able to reproduce the corresponding classical PB for the zero modes; this consistency check has been performed in [1].

Several subtleties appear in this approach. First, the quasiclassical limit of the exchange relations for the quantum zero modes leads to corresponding PB in which the *classical dynamical* r -matrix $r(p)$ has a form which is not consistent with the assumption that $a_C \in G$ for G a simple Lie group. Second, it is known that the classical Yang–Baxter equation (YBE) for the *constant* r -matrix has no solutions for \mathcal{G} (the Lie algebra of G) compact (see, e.g., [8] for comments on this fact).

The first problem has been solved in [1] by assuming that

$$\det a_C = \mathcal{D}_q(p) \neq 1 \quad (1.7)$$

where $\mathcal{D}_q(p)$ is a specific function of the coordinates p of the dual \mathcal{G}^* of the Lie algebra of G (and hence a_C belongs to a reductive extension of the group), and further adding a certain closed p -dependent 2-form to the symplectic form (note that the latter is equivalent to adding to the dynamical r -matrix $r(p)$ a term bilinear in the Cartan generators which is allowed by the Etingof–Varchenko classification [12]). The solution of the second problem can be sought in a similar way, e.g., by using the complexification $G_{\mathbb{C}}$ instead of G itself. Indeed, one is

forced to use G_C when applying Gawędzki's method of constructing a symplectic form for $g_C(x)$ (and a_C) leading to PB with constant r -matrices [14].

The 2D WZNW models corresponding to *compact* G at positive integer levels k belong to the class of *rational* conformal field theories, and this crucial feature is lost when going to simple noncompact or reductive groups. All we have to care about is, however, that $g(x, t) \in G$. Even the chiral components $g_C(x)$ are themselves 'not observable', but in fact condition (1.7) for a_C can be compensated by a suitable renormalization of $u_C(x)$ so we can safely assume that $\det g_C(x) = 1$. One should note that (a special matrix element of) the kernel Q in (1.2) serves, after suitable quantization in an extended space of states, as a generalized BRS operator whose homologies can be used to define a finite-dimensional 'physical' subfactor. In the case of $G = SU(2)$, it has been proved that the dimension of the latter is the right one, $k + 1$ —i.e. equals the number of integrable representations of the affine algebra [17–20]. Thus the standard Hopf algebraic symmetry of the chiral sectors signalled by the presence of constant R -matrices in the exchange relations can be treated in a way reminiscent of the 'covariant gauges' in ordinary gauge theories.

In [1] the $SU(n)_k$ WZNW zero modes' PB were derived by using an extended phase space and a Dirac constraint procedure. This short paper aims to provide an alternative—direct and very elementary—derivation of the WZNW zero modes' PB in the illustrative example of $G = SU(2)$ which allows a convenient parametrization in terms of Euler angles.

The content of the paper is the following. In section 2 we introduce the symplectic form for the chiral $SU(2)_k$ zero modes and compute it in terms of the Euler angle parameters. In section 3 we invert the symplectic form to obtain the PB. The correct classical dynamical r -matrix $r(p)_{12}$ following from the quasiclassical approximation is obtained in section 4. In section 5 the PB of the full monodromy matrix are derived. Finally, in section 6 we discuss the results.

2. Chiral symplectic form for the $SU(2)_k$ zero modes

The symplectic forms for the chiral WZNW zero modes are obtained by keeping only the a_C -dependent parts ($C = L, R$) of the symplectic forms for the chiral fields after using the decomposition $g_C(x) = u_C(x)a_C$ (1.1). We will only consider the left chiral part, denoting from now on $a_L =: a$, $(M_p)_L =: M_p$ (the symplectic form for the right chiral part differs just in sign). One gets [1, 3]

$$\begin{aligned}\Omega(a, M_p) &= \Omega_a(a, M_p) - \frac{k}{4\pi} \rho(a^{-1} M_p a) & \Omega_a(a, M_p) &= \frac{k}{4\pi} \omega(a, M_p) \\ \omega(a, M_p) &= \text{tr}((daa^{-1})(2dM_p M_p^{-1} + M_p(daa^{-1})M_p^{-1})) \\ \rho(a^{-1} M_p a) &= \text{tr}(M_+^{-1} dM_+ M_-^{-1} dM_-) & M_+ M_-^{-1} &= a^{-1} M_p a.\end{aligned}\tag{2.1}$$

The matrices M_+ , M_- are upper, resp. lower triangular and $\text{diag}(M_+) = \text{diag}(M_-^{-1})$.

The 2-form $\Omega(a, M_p)$ is closed, due to

$$d\rho(M) = \theta(M) := \frac{1}{3} \text{tr}(M^{-1} dM)^3 \quad d\omega(a, M_p) = \theta(a^{-1} M_p a).\tag{2.2}$$

Both equations in (2.2) can only be satisfied on properly defined open submanifolds of G ; this is well known [14] and will be made explicit in the calculations below.

In the case $a \in G = SU(2)$, we can parametrize the zero modes⁶ by Euler angles⁶,

$$a = XBA := \begin{pmatrix} e^{i\xi} & 0 \\ 0 & e^{-i\xi} \end{pmatrix} \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}.\tag{2.3}$$

⁶ A similar computation has been performed in [5, 6] for the top. We are considering here the classical ($\hbar = 0$) but deformed (k finite) case.

The diagonal monodromy matrix M_p is given by

$$M_p = \begin{pmatrix} e^{i\frac{\pi}{k}p} & 0 \\ 0 & e^{-i\frac{\pi}{k}p} \end{pmatrix} \equiv q^{-ph} \quad (2.4)$$

where

$$q = e^{-i\frac{\pi}{k}} \quad (2.5)$$

is the (quasi)classical counterpart of the quantum group deformation parameter and

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \sigma_3. \quad (2.6)$$

We have the following expansion of the right invariant Lie algebra-valued zero modes' 1-form:

$$\begin{aligned} -i daa^{-1} &= \sum_{j=1}^3 \Theta^j \sigma_j \\ \Theta^1 &= -\cos 2\xi \sin 2\beta d\alpha + \sin 2\xi d\beta \\ \Theta^2 &= \sin 2\xi \sin 2\beta d\alpha + \cos 2\xi d\beta \\ \Theta^3 &= d\xi + \cos 2\alpha d\alpha \end{aligned} \quad (2.7)$$

where σ_j are the Pauli matrices, and Θ^j are the corresponding basic 1-forms.

The calculation of $\omega(a, M_p)$ is straightforward.

$$\begin{aligned} 2\text{tr}((daa^{-1}) dM_p M_p^{-1}) &= \frac{4\pi}{k} dp(d\xi + \cos 2\beta d\alpha) \\ \text{tr}((daa^{-1}) M_p (daa^{-1}) M_p^{-1}) & \\ &= \text{tr}(B(dAA^{-1})B^{-1}(M_p(dBB^{-1})M_p^{-1} - M_p^{-1}(dBB^{-1})M_p)) \\ &= 4 \sin \frac{2\pi}{k} p \sin 2\beta d\alpha d\beta \end{aligned} \quad (2.8)$$

or, putting the two parts together,

$$\omega(a, M_p) = 4 \left(\sin \frac{2\pi}{k} p \sin 2\beta d\alpha d\beta + \frac{\pi}{k} dp(d\xi + \cos 2\beta d\alpha) \right). \quad (2.9)$$

As expected, $\omega(a, M_p)$ would be real in the compact, $SU(2)$, case.

The full monodromy matrix is given by

$$\begin{aligned} M &= a^{-1} M_p a = M_+ M_-^{-1} \equiv v_0 \mathbb{1} + i \sum_{j=1}^3 v_j \sigma_j \\ &= \begin{pmatrix} \cos \frac{\pi}{k} p + i \cos 2\beta \sin \frac{\pi}{k} p & i e^{-2i\alpha} \sin 2\beta \sin \frac{\pi}{k} p \\ i e^{2i\alpha} \sin 2\beta \sin \frac{\pi}{k} p & \cos \frac{\pi}{k} p - i \cos 2\beta \sin \frac{\pi}{k} p \end{pmatrix}. \end{aligned} \quad (2.10)$$

Obviously, M is unitary and unimodular for real values of the parameters; in particular, the real 4-vector (v_0, v_1, v_2, v_3) given by

$$\left(\cos \frac{\pi}{k} p, \cos 2\alpha \cos 2\beta \sin \frac{\pi}{k} p, \sin 2\alpha \cos 2\beta \sin \frac{\pi}{k} p, \sin 2\beta \sin \frac{\pi}{k} p \right) \quad (2.11)$$

lies on \mathbb{S}^3 .

To compute $\rho(a^{-1} M_p a) = \rho(M_+ M_-^{-1})$, let us parametrize the triangular matrices M_{\pm} as

$$M_+ = \begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix} \quad M_-^{-1} = \begin{pmatrix} z & 0 \\ y & z^{-1} \end{pmatrix} \quad (2.12)$$

where

$$\begin{aligned}xz^{-1} &= i e^{-2i\alpha} \sin \frac{\pi}{k} p \sin 2\beta \\yz^{-1} &= i e^{2i\alpha} \sin \frac{\pi}{k} p \sin 2\beta \\z^{-2} &= e^{-i\frac{\pi}{k}p} \cos^2 \beta + e^{i\frac{\pi}{k}p} \sin^2 \beta = \cos \frac{\pi}{k} p - i \sin \frac{\pi}{k} p \cos 2\beta =: e^{-i\frac{\pi}{k}\delta(p,\beta)} \\ \frac{y}{x} &= e^{4i\alpha} \quad xy z^{-2} = -\sin \frac{2\pi}{k} p \sin^2 2\beta.\end{aligned}\tag{2.13}$$

Then we can express $\rho(M_+ M_-^{-1})$ as

$$\begin{aligned}\rho(M_+ M_-^{-1}) &= z^{-2} (dy dx + z^{-1} dz (y dx - x dy)) \\ &= \frac{1}{2} d(xyz^{-2}) d \log \frac{y}{x} - z^{-2} \left(d(xz^{-1}) d \frac{yz^{-1}}{z^{-2}} + d \frac{xz^{-1}}{z^{-2}} d(yz^{-1}) \right).\end{aligned}\tag{2.14}$$

This gives

$$\rho(a^{-1} M_p a) = 4i \left(\sin \frac{\pi}{k} p \sin 2\beta \right)^2 d\alpha d \log \frac{\cos \frac{\pi}{k} p - i \sin \frac{\pi}{k} p \cos 2\beta}{\sin \frac{\pi}{k} p \sin 2\beta}.\tag{2.15}$$

Note that the 2-form ρ is not real. It is easy to check, however, that the *external derivatives* of both ω and ρ are equal to the *real* canonical (Wess–Zumino) 3-form θ which coincides, in this case, with the volume form on $SU(2)$ [7]. The equality of $d\omega$ and $d\rho$ ensures that $\Omega(a, M_p)$ is closed:

$$d\omega = d\rho = -\frac{16\pi}{k} \sin \frac{2\pi}{k} p \sin 2\beta d\alpha d\beta dp \equiv \theta \quad \Rightarrow \quad d\Omega(a, M_p) = 0.\tag{2.16}$$

The full symplectic form becomes very simple in this (complex) parametrization:

$$\Omega(a, M_p) = d(p d\xi + \delta(p, \beta) d\alpha).\tag{2.17}$$

Hence, (ξ, p) and $(\alpha, \delta(p, \beta))$, where

$$\delta(p, \beta) = i \frac{k}{\pi} \log \left(\cos \frac{\pi}{k} p - i \sin \frac{\pi}{k} p \cos 2\beta \right)\tag{2.18}$$

form canonical pairs. This fact provides an independent explanation of the chosen normalization of the matrix elements of M_p in (2.4) and of $\delta(p, \beta)$ in (2.13) (the original motivation being the quasiclassical correspondence [1]).

3. Poisson brackets

Inverting the symplectic form, one obtains the Poisson brackets. To get the elementary ones for the basic phase space variables, $(q^i) := (\xi, p, \alpha, \beta)$, we present $\Omega(a, M_p)$ as

$$\Omega(a, M_p) = \frac{1}{2} \omega_{ij} dq^i dq^j \quad \Rightarrow \quad \{q^i, q^j\} = \omega^{ij} \quad \omega^{ij} \omega_{j\ell} = \delta_\ell^i.\tag{3.19}$$

The result for the nonzero PB is

$$\begin{aligned}\{\xi, p\} &= 1 \\ \{\alpha, \beta\} &= \frac{\pi}{k} \frac{i \cos 2\beta \sin \frac{\pi}{k} p - \cos \frac{\pi}{k} p}{2 \sin 2\beta \sin \frac{\pi}{k} p} \\ \{\xi, \beta\} &= \frac{\pi}{k} \frac{\cos 2\beta \cos \frac{\pi}{k} p - i \sin \frac{\pi}{k} p}{2 \sin 2\beta \sin \frac{\pi}{k} p}.\end{aligned}\tag{3.20}$$

Now using the Leibniz rule we can obtain the PB for the matrix elements of

$$a = \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix} = \begin{pmatrix} e^{i(\xi+\alpha)} \cos \beta & e^{i(\xi-\alpha)} \sin \beta \\ -e^{i(\alpha-\xi)} \sin \beta & e^{-i(\xi+\alpha)} \cos \beta \end{pmatrix} \quad (3.21)$$

the six independent ones of which are

$$\begin{aligned} \{a_1^1, a_2^1\} &= \frac{\pi}{k} e^{2i\xi} \sin \beta \cos \beta = \frac{\pi}{k} a_2^1 a_1^1 \\ \{a_1^2, a_2^2\} &= -\frac{\pi}{k} e^{-2i\xi} \sin \beta \cos \beta = \frac{\pi}{k} a_2^2 a_1^2 \\ \{a_1^1, a_1^2\} &= i \frac{\pi}{k} e^{i\alpha} \cot \frac{\pi}{k} p \sin \beta \cos \beta = -i \frac{\pi}{k} \cot \frac{\pi}{k} p a_1^2 a_1^1 \\ \{a_2^1, a_2^2\} &= -i \frac{\pi}{k} e^{-i\alpha} \cot \frac{\pi}{k} p \sin \beta \cos \beta = -i \frac{\pi}{k} \cot \frac{\pi}{k} p a_2^2 a_2^1 \\ \{a_1^1, a_1^2\} &= -\frac{\pi}{k} \cos^2 \beta \left(1 + i \cot \frac{\pi}{k} p\right) = -\frac{\pi}{k} a_1^1 a_2^2 - i \frac{\pi}{k} \cot \frac{\pi}{k} p a_2^2 a_1^1 \\ \{a_2^1, a_2^2\} &= -\frac{\pi}{k} \sin^2 \beta \left(1 - i \cot \frac{\pi}{k} p\right) = \frac{\pi}{k} a_2^1 a_1^2 - i \frac{\pi}{k} \cot \frac{\pi}{k} p a_1^2 a_2^1. \end{aligned} \quad (3.22)$$

Using the standard compact tensor product notations $a_1 = a \otimes \mathbb{1}$, $a_2 = \mathbb{1} \otimes a$ etc, one can conveniently present the above PB in the form

$$\{a_1, a_2\} = r^\#(p)_{12} a_1 a_2 - a_1 a_2 \frac{\pi}{k} r_{12} \quad (3.23)$$

where

$$r_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad r^\#(p)_{12} = -i \frac{\pi}{k} \cot \frac{\pi}{k} p r_{12} \quad (3.24)$$

are the *constant* and the *dynamical* skewsymmetric r -matrices, respectively. Note that the imaginary unit in the expression for $r^\#(p)$ makes the latter expressible, with real coefficients, in terms of the *compact* ($\mathcal{G} = su(2)$) generators,

$$r^\#(p) = \frac{2\pi}{k} \cot \frac{\pi}{k} p (\tau_1 \otimes \tau_2 - \tau_2 \otimes \tau_1) \quad 2i\tau_j = \sigma_j \quad [\tau_i, \tau_j] = \varepsilon_{ij\ell} \tau_\ell \quad (3.25)$$

($i, j, \ell = 1, 2, 3$), i.e. $r^\#(p) \in \mathcal{G} \otimes \mathcal{G}$, whereas the constant r -matrix belongs instead to $\mathcal{G}_{\mathbb{C}} \otimes \mathcal{G}_{\mathbb{C}} = sl(2) \otimes sl(2)$,

$$r = e \otimes f - f \otimes e \quad e = i\tau_1 - \tau_2 \quad f = i\tau_1 + \tau_2 \quad [e, f] = h. \quad (3.26)$$

The Jacobi identity of the PB (3.23) is guaranteed due to the modified classical YBE satisfied by r_{12} , resp. the modified classical dynamical YBE satisfied by $r^\#(p)_{12}$; as mentioned in the introduction, the modified classical YBE has no constant solutions for compact simple Lie algebras [8].

The PB between the entries of M_p and a can be obtained immediately, the only nontrivial relations being those between the (entries of the) diagonal matrices M_p and X :

$$\{M_{p1}, a_2\} = \frac{2\pi}{k} \sigma_{12} M_{p1} a_2 \quad \sigma := \frac{1}{2} \sigma_3 \otimes \sigma_3. \quad (3.27)$$

Note that σ is just the diagonal of the polarized Casimir operator matrix,

$$C = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h \quad (3.28)$$

and that $r^\#(p)_{12}$ obeys the equation

$$(Ad(M_{p1}) - 1) r^\#(p)_{12} = \frac{\pi}{k} (Ad(M_{p1}) + 1) (C_{12} - \sigma_{12}). \quad (3.29)$$

Obviously, $\{M_{p1}, M_{p2}\} = 0$.

4. Changing the dynamical r -matrix

The PB of the classical model described above should appear as a quasiclassical limit of the corresponding quantum exchange algebra [10]. This limit (see, e.g., [1]) essentially means that one only retains the first order in the $1/k$ asymptotic expansion of the quantum R -matrices and replaces the commutators by Poisson brackets applying Dirac’s quantization principle backwards. The quantum dynamical R -matrix $R(p)_{12}$ has been obtained independently, for $G = SU(n)$, from the braiding properties of the suitable conformal blocks [21] in [16] and as a solution of the quantum dynamical YBE—in [22]; the corresponding ‘quantum matrix algebra of $SL(n)$ -type’ (the exchange algebra appearing as a quantized version of the chiral zero modes) has been studied in detail in [13].

It has been shown in [1] that the correct classical dynamical r -matrix for $G = SU(n)$ (of size $n^2 \times n^2$) obtained this way is given by

$$r(p)_{j'\ell'}^{j\ell} = \begin{cases} i\frac{\pi}{k} \cot \frac{\pi}{k} p_{j\ell} (\delta_{j'}^j \delta_{\ell'}^{\ell} - \delta_{\ell'}^j \delta_{j'}^{\ell}) & \text{for } j \neq \ell \\ 0 & \text{for } j = \ell. \end{cases} \quad (4.30)$$

Here $p_{j\ell} = p_j - p_\ell$, $j, \ell = 1, \dots, n$, are coordinates in the dual space $\mathcal{G}_{\mathbb{C}}^*$ of the Lie algebra $\mathcal{G}_{\mathbb{C}}$ so that nonnegative integer values of the $(n-1)$ independent of them, p_{jj+1} , correspond to the $s\ell(n)$ dominant weights. Comparing, for $n = 2$ and $p = p_{12}$, (4.30) with (3.24), one sees that our $r^\#(p)_{12}$ only reproduces the non-diagonal entries of the correct classical dynamical r -matrix $r(p)_{12}$.

The solution of this problem has been found in [1]. First, one observes that the symplectic form $\Omega(a, M_p)$ (2.1) is invariant w.r.t. rescaling of $a \rightarrow f(p)a$ with $f(p)$ a scalar function (this does not change the monodromy). It is easy to see that the PB for the rescaled zero mode matrix elements get additional terms amounting to adding diagonal terms to $r^\#(p)_{12}$. In the general $SU(n)$ case, to recover exactly (4.30), one also needs to add a closed p -dependent 2-form to $\Omega(a, M_p)$; in the $n = 2$ case such a form does not exist since the differential algebra on $p_{j\ell}$, $1 \leq j < \ell \leq 2$, is of course one dimensional.

In [1] the computations have been made using an ‘extended’ phase space on which $\det a$ and $P := \frac{1}{n}(p_1 + \dots + p_n)$ are not subject to any other conditions except regularity, $\det a \neq 0$. A Dirac reduction on the submanifold

$$\det a = \mathcal{D}_q(p) \quad P = 0 \quad (4.31)$$

where

$$\mathcal{D}_q(p) := \prod_{i < j}^n [p_{ij}] \quad [p] := \frac{q^p - q^{-p}}{q - q^{-1}} \quad (4.32)$$

for q given by (2.5), was then shown to lead to the quasiclassically expected PB and, in particular, to the correct classical dynamical r -matrix.

Let us compute in the $n = 2$ case the PB of the rescaled zero modes

$$a \rightarrow [p]^{\frac{1}{2}} a. \quad (4.33)$$

Remark. The operation (4.33) is not innocent since, for $p \in k\mathbb{Z}$, the bracket $[p]$ vanishes. In fact, at these values of p , the function $\cot \frac{\pi}{k} p$ is not defined as well. Therefore, we must exclude them from the outset, thus restricting the possible values of the diagonal monodromy (2.4). It is easy to see that the forbidden values of p are exactly those for which M_p as well as M , the full monodromy matrix (2.10), belongs to the centre of G (or $G_{\mathbb{C}}$).

The only nontrivial PB of the scaling factor is with the ξ variable,

$$\{\xi, [p]^{\frac{1}{2}}\} = \frac{\pi}{2k} [p]^{\frac{1}{2}} \cot \frac{\pi}{k} p \quad (4.34)$$

and using (3.20) and (4.34), one obtains, for example,

$$\begin{aligned} \{a_1^1, a_2^2\} &= \{[p]^{\frac{1}{2}} e^{i(\xi+\alpha)} \cos \beta, [p]^{\frac{1}{2}} e^{-i(\xi+\alpha)} \cos \beta\} \\ &= 2i \cos^2 \beta [p]^{\frac{1}{2}} \{\xi, [p]^{\frac{1}{2}}\} + [p] \{e^{i(\xi+\alpha)} \cos \beta, e^{-i(\xi+\alpha)} \cos \beta\} \\ &= \frac{\pi}{k} a_2^1 a_1^2 + i \frac{\pi}{k} \cot \frac{\pi}{k} p (a_1^1 a_2^2 - a_1^2 a_2^1). \end{aligned} \quad (4.35)$$

The full list of the modified PB for the zero modes can be compactly written as

$$\begin{aligned} \{a_\alpha^j, a_\beta^j\} &= -\frac{\pi}{k} a_\alpha^j a_\beta^j \epsilon_{\alpha\beta} \quad j, \alpha, \beta = 1, 2 \quad (\epsilon_{\alpha\beta}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \{a_\alpha^1, a_\beta^2\} &= -\frac{\pi}{k} \left(i \cot \frac{\pi}{k} p (\det a) + a_\alpha^2 a_\beta^1 \right) \epsilon_{\alpha\beta} \\ &= -\frac{\pi}{k} \left(i \frac{\cos \frac{\pi}{k} p}{\sin \frac{\pi}{k}} + a_\alpha^2 a_\beta^1 \right) \epsilon_{\alpha\beta} \end{aligned} \quad (4.36)$$

(no summation over α and β is assumed). The PB (4.36) are of the form

$$\{a_1, a_2\} = r(p)_{12} a_1 a_2 - a_1 a_2 \frac{\pi}{k} r_{12} \quad (4.37)$$

with

$$r(p) = r^\#(p) + \frac{2\pi}{k} \cot \frac{\pi}{k} p (\mathbb{1} \otimes \tau_3 - \tau_3 \otimes \mathbb{1}) \quad (4.38)$$

($r^\#(p)$ is given by (3.25)). The matrix

$$r(p)_{12} = i \frac{\pi}{k} \cot \frac{\pi}{k} p \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.39)$$

coincides with (4.30) for $n = 2$, $p_{12} := p = -p_{21}$.

Obviously, equation (3.27) does not change after the rescaling.

5. Poisson brackets for the full monodromy matrix M

The PB of the zero modes with the full monodromy matrix M as well as the PB between the matrix elements of M should not contain the dynamical r -matrix. This conclusion follows immediately from the quasiclassical limit of the corresponding quantum exchange algebra which does not contain the quantum dynamical R -matrix, see, e.g., [10]. On the other hand, using $M = a^{-1} M_p a$ (2.10), one should be able to reproduce this directly. Indeed,

$$\begin{aligned} \{M_1, a_2\} &= \{a_1^{-1} M_{p1} a_1, a_2\} \\ &= -a_1^{-1} \{a_1, a_2\} a_1^{-1} M_{p1} a_1 + a_1^{-1} \{M_{p1}, a_2\} a_1 + a_1^{-1} M_{p1} \{a_1, a_2\} \\ &= \frac{\pi}{k} a_2 r_{12} M_1 - a_1^{-1} r(p)_{12} M_{p1} a_1 a_2 + \frac{2\pi}{k} a_1^{-1} \sigma_{12} M_{p1} a_1 a_2 \\ &\quad + a_1^{-1} M_{p1} r(p)_{12} a_1 a_2 - \frac{\pi}{k} M_1 a_2 r_{12} \\ &= \frac{\pi}{k} a_2 (r_{12} M_1 - M_1 r_{12}) + a_1^{-1} \left(M_{p1} r(p)_{12} - r(p)_{12} M_{p1} + \frac{2\pi}{k} \sigma_{12} M_{p1} \right) a_1 a_2. \end{aligned} \quad (5.40)$$

Now note that $r(p)$ (4.38) obeys (3.29) together with $r^\#(p)$ since the diagonal elements lie in the kernel of the operator $Ad(M_{p1}) - 1$ (this means, in particular, that the present derivation is valid for both dynamical r -matrices); hence,

$$M_{p1}r(p)_{12} - r(p)_{12}M_{p1} + \frac{2\pi}{k}\sigma_{12}M_{p1} = \frac{\pi}{k}(M_{p1}C_{12} + C_{12}M_{p1}). \quad (5.41)$$

We have, therefore,

$$\begin{aligned} \{M_1, a_2\} &= \frac{\pi}{k}a_2(r_{12}M_1 - M_1r_{12}) + \frac{\pi}{k}a_1^{-1}(M_{p1}C_{12}a_1a_2 + C_{12}M_{p1}a_1a_2) \\ &= \frac{\pi}{k}a_2(r_{12}M_1 - M_1r_{12}) + \frac{\pi}{k}a_1^{-1}(M_{p1}a_1a_2C_{12} + a_1a_2C_{12}a_1^{-1}M_{p1}a_1) \\ &= \frac{\pi}{k}a_2(r_{12}M_1 - M_1r_{12}) + \frac{\pi}{k}a_2(M_1C_{12} + C_{12}M_1) \\ &= \frac{\pi}{k}a_2(r_{12}^+M_1 - M_1r_{12}^-) \end{aligned} \quad (5.42)$$

(we have used the basic property of the polarized Casimir, $[C_{12}, a_1a_2] = 0$), where

$$r_{12}^\pm := r_{12} \pm C_{12} \quad (5.43)$$

i.e. we get the relation of the type we expect.

Note that the first relation in (3.27) is equivalent to

$$\{a_1, M_{p2}\} = -\frac{2\pi}{k}\sigma_{12}a_1M_{p2}. \quad (5.44)$$

Hence,

$$\begin{aligned} \{M_1, M_{p2}\} &= \{a_1^{-1}M_{p1}a_1, M_{p2}\} \\ &= -a_1^{-1}\{a_1, M_{p2}\}a_1^{-1}M_{p1}a_1 + a_1^{-1}M_{p1}\{a_1, M_{p2}\} \\ &= \frac{2\pi}{k}a_1^{-1}\sigma_{12}a_1M_{p2}a_1^{-1}M_{p1}a_1 - \frac{2\pi}{k}a_1^{-1}M_{p1}\sigma_{12}a_1M_{p2} \\ &= \frac{2\pi}{k}a_1^{-1}(\sigma_{12}M_{p2}M_{p1} - M_{p1}\sigma_{12}M_{p2})a_1 = 0 \end{aligned} \quad (5.45)$$

(M_{p1} , M_{p2} and σ_{12} are all diagonal and hence, commute).

We can obtain, finally, the relation for the PB $\{M_1, M_2\}$ as well:

$$\begin{aligned} \{M_1, M_2\} &= \{M_1, a_2^{-1}M_{p2}a_2\} \\ &= -a_2^{-1}\{M_1, a_2\}a_2^{-1}M_{p2}a_2 + a_2^{-1}M_{p2}\{M_1, a_2\} \\ &= M_2a_2^{-1}\{M_1, a_2\} - a_2^{-1}\{M_1, a_2\}M_2 \\ &= \frac{\pi}{k}(M_2(r_{12}^+M_1 - M_1r_{12}^-) - (r_{12}^+M_1 - M_1r_{12}^-)M_2) \\ &= -\frac{\pi}{k}(M_1M_2r_{12} + r_{12}M_1M_2 - M_2r_{12}^+M_1 - M_1r_{12}^-M_2). \end{aligned} \quad (5.46)$$

6. Discussion and outlook

Although the purpose of this work is to be as explicit as possible, many of the computations and the comments are subject to immediate generalization (e.g., for higher compact Lie groups). This concerns, for example, equation (3.29), which can be derived, in general, by solving the equations for the Hamiltonian vector fields (see [14]). For $G = SU(2)$ the solutions (3.25) and (4.38) of equation (3.29) (both satisfying, in addition, the modified classical dynamical YBE) correspond to different choices for a —in the first case a belongs to G itself and in the

second, to a reductive extension of G . As shown in [1], for $SU(n)$, $n > 2$, one also needs to add a closed p -dependent form to the chiral symplectic form in order to get the ‘correct’ $r(p)_{12}$ (i.e. the one following from the quasiclassical correspondence, given by (4.38) for $n = 2$).

What characterizes the WZNW PB with *constant* r -matrices is their Poisson–Lie invariance (see, e.g., [4]) which is the classical counterpart of the quantum group invariance of the corresponding exchange relations for the chiral fields $g_C(x)$ [14]. The price one has to pay for this (relatively simple—in particular, coassociative) symmetry is the necessity of extending the original WZNW phase space. As mentioned in the introduction, the main open problem is to provide a general procedure for finding the proper ‘physical quotient’ of this extension. Qualitatively different is the situation for the chiral fields $u_C(x)$ with diagonal monodromy whose quantized counterparts are the chiral vertex operators [10]. Their PB involve the dynamical r -matrix $r(p)_{12}$ in place of r_{12} . For a third possibility—of defining a *quasi-Poisson* structure on the chiral WZNW phase space—see, e.g., the discussion in [15] (and references therein). It leads, upon quantization, to a quasi-Hopf (hence, non-coassociative) deformed symmetry [23]. In all these cases the corresponding exchange algebras encode the monodromy properties of the solutions of the Knizhnik–Zamolodchikov equation for the WZNW conformal blocks.

Clearly, the zero modes contain the whole information about the generalized WZNW internal symmetry acting from the right on the chiral fields, cf (1.1). They form a finite-dimensional dynamical system and hence are easier to study. The present paper is an attempt to providing some more details clarifying the structure of the classical model.

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